Sequence of phase transitions induced in an array of Josephson junctions by their crossover to π-state

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Abstract. - We show that the transition of Josephson junctions between the conventional and π states caused by the decrease in temperature induces in a regular two-dimensional array of such junctions not just a single phase transition between two phases with different ordering but a sequence of two, three or four phase transitions. The corresponding phase diagrams are constructed for the cases of bipartite (square or honeycomb) and triangular lattices.

Introduction. – For several decades arrays of weakly coupled superconducting islands have been the subject of active experimental investigations [1] for many reasons, in particular as a simple model system which allows one to study the interplay between fluctuations, frustration, disorder and other factors in a more controlled situation than in bulk superconductors. However, these studies have been restricted to arrays of conventional junctions whose energy is minimal when the phases of two superconductors are equal to each other.

The first experimental realization of an old theoretical idea [2, 3] about fabrication of so-called π-junction whose energy is minimal when the phase difference on the junction is equal to π was achieved only during last decade by Ryazanov *et al.* [4] who studied superconductor-ferromagnet-superconductor (SFS) Josephson junctions and observed a transition from the conventional state to the π-state taking place with the decrease in temperature [3]. The experimental investigation of small arrays of SFS junctions started almost simultaneously [5], but insofar has been restricted to very modest sizes [6].

Since the fabrication of more sizable arrays of SFS junctions is definitely a matter of the nearest future, the present letter addresses the question what happens with a superconducting array of Josephson junctions when the decrease in temperature induces a crossover of the junctions to the π-state. Although one could expect (from the evident change of the ground state structure) that this induces a single first-order transition between two phases with different ordering, our analysis reveals that this is

never the case and in reality an array experiences in the crossover region a sequence of two, three or even four phase transitions each of which is related with partial or complete destruction (or restoration) of ordering. The structures of phase diagrams and the natures of these transitions are established both for bipartite lattices (square and honeycomb) and for a triangular one.

Model. – An array of identical SFS junctions can be described by the Hamiltonian

$$H = \sum_{\langle \mathbf{jj'} \rangle} V(\varphi_{\mathbf{j}} - \varphi_{\mathbf{j'}}), \tag{1}$$

where $\varphi_{\mathbf{j}}$ is the phase of the superconducting order parameter on $\mathbf{j}$th superconducting island, the summation is performed over all pairs of neighboring islands connected by a junction and $V(\theta)$ is a periodic even function of θ which can have minima both at $\theta = 0$ and $\theta = \pi$. When the contacts forming a junction have low transparency, one can keep in the Fourier expansion of

$$V(\theta) = -\sum_{p=1}^{\infty} J_p \cos(p\,\theta)$$

only the first term because a typical value of J_p is strongly suppressed with the increase of p [3].

However, in a SFS junction of an appropriate width the decrease in temperature T may force the value of J_1 to pass through zero and change sign [7]. This leads to the transition of the junction from the conventional state [in which the deepest minimum of $V(\theta)$ is at $\theta = 0$] to the π-state (in

which the deepest minimum is at $\theta = \pi$). Naturally, in the vicinity of T_0, the temperature at which $J_1(T) = 0$, one has to keep also the next term in the Fourier expansion of $V(\theta)$,

$$V(\theta) = -J_1 \cos\theta - J_2 \cos(2\theta). \qquad (2)$$

In the simplest situation the decrease of T leads to the change of $J_1(T)$ from positive to negative, while $J_2(T)$ remains positive. Our aim consists in analyzing what phase transition (or what sequence of phase transitions) takes place in a regular array of identical SFS junctions when they experience such a transition to the π-state (also known as $0 - \pi$ crossover).

Bipartite lattice. – First one has to understand what would take place with the change of the sign of J_1 in the absence of thermal fluctuations. For $J_{1,2} > 0$ the minimum of the Hamiltonian (1) with interaction (2) on any lattice is achieved when all variables $\varphi_{\mathbf{j}}$ (defined modulo 2π) are equal to each other, $\varphi_{\mathbf{j}} = \Phi$. Therefore, the ground state is characterized by $U(1)$ degeneracy related to the simultaneous rotation of all phases.

The form of the ground state at $J_1 < 0$ depends on the structure of the lattice. We start by considering the case of a bipartite lattice (square or honeycomb) and after that will discuss the more complex case of a triangular lattice. For any bipartite lattice the problem with $J_1 < 0$ can be mapped onto the problem with $J_1 > 0$ just by rotating half of the variables $\varphi_{\mathbf{j}}$ by π. In particular, this immediately defines the form of the ground state at $J_1 < 0$, which has the same $U(1)$ degeneracy as at $J_1 > 0$ but a different (two-sublattice) structure.

When $J_1 = 0$, the energies of these two states are equal to each other, as well to the energy of any state in which all variables $\varphi_{\mathbf{j}}$ are equal either to Φ or to $\Phi + \pi$. Therefore, in the absence of thermal fluctuations the system would experience at $J_1 = 0$ a single phase transition between the phases with different ordering. Note that this property is not the consequence of keeping only two terms in Eq. (2) - for a more complex form of $V(\theta)$ the transition will be shifted from the point where $J_1 = 0$ to the point where the two minima of $V(\theta)$ have equal depths. However, it turns out that in the presence of thermal fluctuations the single-transition scenario does not survive.

The finite temperature phase diagram of the XY model with a modified Berezinskii-Villain interaction whose main features are analogous to those of Eq. (2) with $J_1, J_2 > 0$ has been constructed in Refs. [8]. In terms of the SFS array problem with interaction (2) the main conclusions of these works (confirmed in numerical simulations of Ref. [9]) can be reformulated and generalized as follows.

When both J_1 and J_2 are positive and much larger than T, the system is in the phase with an algebraic decay of the correlation function

$$C_1(\mathbf{j}_1 - \mathbf{j}_2) = \langle \exp i(\varphi_{\mathbf{j}_1} - \varphi_{\mathbf{j}_2}) \rangle. \qquad (3)$$

For brevity we shall call this phase ferromagnetic, although more accurately it should be called a phase with al-

gebraically decaying ferromagnetic correlations. But since in two-dimensional systems with a continuous order parameter the real long-range order is impossible [10] and an algebraic decay of correlations [11] is as much as one can get, the application of such a shorthand is rather natural. In terms of SFS array this phase is superconducting and is characterized by a finite superfluid density.

The decrease of J_1 down to $J_1 \sim T$ induces a phase transition of the Ising type related to the proliferation of solitons (a soliton is a linear topological excitation on crossing which the phase jumps by π). The existence of such a transition is especially evident for $J_2 = \infty$ when the model defined by Eqs. (1) and (2) is reduced to the Ising model with coupling constant J_1, however it exists (and has the same nature) also when J_2 is less than infinite. The proliferation of solitons leads to the replacement of the algebraic decay of the correlation function $C_1(\mathbf{r})$ by an exponential one. On the other hand, on both sides of the transition the superfluid density remains finite, which for $J_2 < \infty$ manifests itself in the algebraic decay of the correlation function

$$C_2(\mathbf{j}_1 - \mathbf{j}_2) = \langle \exp 2i(\varphi_{\mathbf{j}_1} - \varphi_{\mathbf{j}_2}) \rangle. \qquad (4)$$

It is clear that in the phase with such a behavior of $C_1(\mathbf{r})$ and $C_2(\mathbf{r})$ the role of the order parameter is played by $\exp(2i\varphi_{\mathbf{j}})$ and therefore formally it can be called nematic. Analogous nematic phase (induced by the proliferation of solitons) is expected to exist in thin films of superfluid ^{3}He [12]. In the nematic phase of a SFS array, the superconducting current can be associated with the motion of pairs of Cooper pairs and therefore this phase can be identified by studying the periodicity of the persistent current in the array with annular geometry penetrated by a magnetic flux (the period has to be equal to half of the superconducting flux quantum).

The relevant topological excitations in the nematic phase are halfvortices, that is the vortices with topological charges $\pm 1/2$ which are the end points of solitons. The interaction of these objects is logarithmic and keeps them bound in pairs, which allows one to treat solitons as closed lines playing the role of domain walls in the Ising model. With decrease in J_2 the strength of the logarithmic interaction of half-vortices goes down and at $J_2 \sim T$ it becomes too weak to keep them bound in pairs. The phase transition related to the dissociation of bound pairs of half-vortices is of the Berezinskii-Kosterlitz-Thouless (BKT) type. It differs from the standard BKT transition by the value of the superfluid density jump, which is larger by the factor of 4. In the disordered phase the superfluid density vanishes and correlation function $C_2(\mathbf{r})$ also decays exponentially.

For $J_2 \ll T$ the disordered phase is separated from the ferromagnetic phase existing at large enough ratio J_1/T by the standard BKT transition related with the dissociation of pairs of integer vortices (exactly like at $J_2 = 0$). With the decrease in the ratio T/J_2 one encounters a tricritical point, where this BKT transition is transformed into a
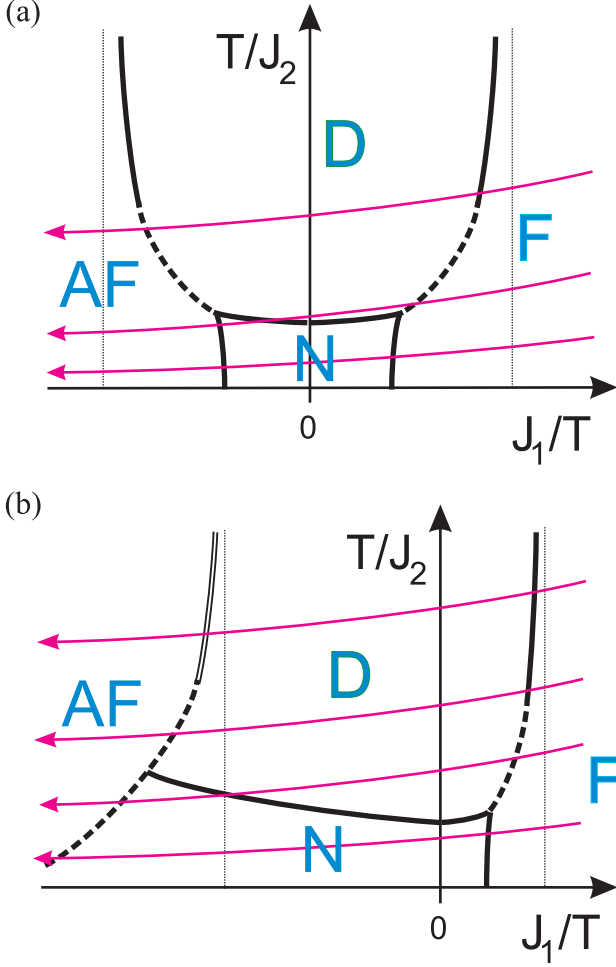
(a)

(b)

Fig. 1: (Color on-line) Schematic structures of phase diagrams of SFS arrays with (a) bipartite lattice and (b) triangular lattice. Ferromagnetic (F), nematic (N), antiferromagnetic (AF) and disordered (D) phases are separated from each other either by continuous (continuous bold lines) or first-order (dashed bold lines) phase transitions. The double line separating AF and D phases in (b) stands for the sequence of BKT and Ising transitions with very small separation. Curved arrows going from right to left show different paths of the evolution of an array with the decrease in temperature.

first-order one (with larger than universal jump of the superfluid density). A change in the nature of the transition can be associated with switching on of a different mechanism for the destruction of the ferromagnetic ordering. On the other side of the tricritical point the disordering is triggered not by the integer vortex pair unbinding but by the proliferation of solitons taking place when the logarithmic interaction of half-vortices is too weak to keep them bound in pairs. This induces the simultaneous unbinding of integer vortices which takes place not because their direct logarithmic interaction is insufficiently strong but because it is screened by the presence of free half-vortices.

The schematic structure of the phase diagram containing ferromagnetic (F), nematic (N) and disordered (D)

phases is shown in Fig. 1(a) in coordinates J_1/T and T/J_2. Although the above analysis refers only to the right half of this figure (with $J_1/T > 0$), in the case of a SFS array with a bipartite lattice it is clear from the symmetry of the problem that at negative values of J_1/T the phase diagram has exactly the same form as at positive, the only difference being that the phase with the ferromagnetic algebraic correlations is replaced by the phase with the antiferromagnetic algebraic correlations (which have the two-sublattice structure).

The evolution of a SFS array with the decrease in temperature is shown in Fig. 1(a) by curved arrows going from right to left. From the structure of the phase diagram it is clear that when thermal fluctuations are taken into account the direct phase transition between the ferromagnetic and antiferromagnetic phases is no longer possible and is replaced by a finite region containing either one or two intermediate phases.

In particular, for sufficiently low values of $T_0/J_2(T_0)$ the evolution goes along the path F-N-AF, that is, the ferromagnetic and antiferromagnetic phases are separated by the nematic phase, both phase transitions being of the Ising type. On the other hand, for sufficiently high values of $T_0/J_2(T_0)$ the ferromagnetic and antiferromagnetic phases are separated by the strip of the disordered phase and the phase transitions are either of the BKT type or of the first order. For intermediate values of $T_0/J_2(T_0)$ the evolution has to take place along the path F-D-N-AF involving three different phase transitions and if in the region where $|J_1(T)|$ is comparable with T or smaller the ratio $T/J_2(T)$ changes extremely little (by less than few percent), the path F-N-D-N-AF involving *four* phase transitions is also possible, although it hardly can be called a typical one.

Triangular lattice. – In the case of a triangular lattice the structure of the phase diagram at $J_1 > 0$ is basically the same as for a bipartite lattice, whereas at $J_1 < 0$ the situation is essentially different. The main reason for that is that at negative J_1 the structure of the ground state is different for small and for large values of $|J_1|$. In particular, for $-9J_2 < J_1 < 0$ the minimum of energy is achieved when on each triangular plaquette the phase difference on two bonds is equal to π and on the third one to zero. It is clear that in any configuration satisfying this rule the variables $\varphi_{\mathbf{j}}$ can acquire only two values which differ by π (for example, Φ and $\Phi + \pi$), from where it follows that in terms of the nematic order parameter $\exp(2i\varphi_{\mathbf{j}})$ the system is perfectly ordered.

After introducing bimodal variables $\sigma_{\mathbf{j}} = \pm 1$ (below they are called pseudospins) such that

$$\exp(i\varphi_{\mathbf{j}}) = \exp(i\Phi)\sigma_{\mathbf{j}}\,, \qquad (5)$$

one finds that the above-mentioned rule is satisfied as soon as each triangular plaquette contains both positive and negative pseudospins. This means that the set of the allowed configurations of pseudospins $\sigma_{\mathbf{j}}$ coincides with the

set of the ground states of the antiferromagnetic Ising model with triangular lattice (the AFMITL model). The number of such configurations grows exponentially with the size of the system [13]. The exact solution of the AFMITL model [13, 14] at zero temperature is characterized by an algebraic decay of the correlation functions [15], in particular, $\langle \sigma_{\mathbf{j}_1} \sigma_{\mathbf{j}_2} \rangle \propto |\mathbf{j}_1 - \mathbf{j}_2|^{-1/2}$. These correlations have the three-sublattice antiferromagnetic structure, that is are positive when the two pseudospins belong to the same triangular sublattice and negative otherwise [15]. From the form of Eq. (5) it is then clear that at zero temperature $C_1(\mathbf{j}_1 - \mathbf{j}_2)$ coincides with $\langle \sigma_{\mathbf{j}_1} \sigma_{\mathbf{j}_2} \rangle$ and therefore has a three-sublattice antiferromagnetic structure.

At $J_1 < -9J_2$ the ground state of (1) has exactly the same structure as at $J_2 = 0$. In this state each of the three sublattices is ferromagnetically ordered but the phases in the different sublattices are rotated with respect to each other by $\pm 2\pi/3$ [16]. The full set of ground states is characterized by a combined $U(1) \times Z_2$ degeneracy, where $U(1)$ corresponds to the simultaneous rotation of all phases and Z_2 can be associated with antiferromagnetic ordering of chiralities of triangular plaquettes. Thus in the absence of thermal fluctuations the phase diagram of a SFS array with triangular lattice would incorporate three different phases, the phases with ferromagnetic and antiferromagnetic ordering being separated by a wide strip of the phase with perfect nematic ordering and an algebraic decay of antiferromagnetic correlations.

At finite temperatures the perfect antiferromagnetic ordering existing at $J_1 < -9J_2$ is naturally replaced by an algebraic decay of $C_1(\mathbf{r})$, however a finite superfluid density and the genuine long-range order in staggered chirality survive. It is known both from numerical simulations [17] and analytical considerations [18] that at $J_2 = 0$ the disordering of the system with the increase in temperature takes place through the sequence of two phase transitions which are situated very close to each other. The first of them is related to vortex pairs dissociation and is of the BKT type, whereas the second is related with domain wall proliferation and is of the Ising type. It follows from the analysis of the mutual influence of the topological excitations of different types [18] that the same scenario can be expected to hold also when $J_2 > 0$.

The properties of the nematic phase are influenced by a small finite temperature more drastically than that of the antiferromagnetic phase. It is known both from the exact solutions [13, 14] and from the mapping onto a solid-on-solid (SOS) model [19] that at any finite temperature the isotropic AFMITL model is in the disordered phase with a finite correlation radius (which diverges when $T \to 0$). This immediately allows one to conclude that at $T > 0$ the nematic phase is characterized by an exponential decay of $C_1(\mathbf{r})$. On the other hand, spin wave fluctuations lead to an algebraic decay of $C_2(\mathbf{r})$. These properties are in perfect agreement with those of the nematic phase at $J_1 > 0$, which is no surprise since this is just the same phase. Exactly like at $J_1 > 0$, at $J_1 < 0$ the nematic phase

is characterized by a finite superfluid density and its disordering takes place via BKT phase transition related to the dissociation of halfvortex pairs. One more example of an XY model in which the phase transition into a disordered phase is related to the dissociation of halfvortex pairs is the frustrated XY model with dice lattice and one-third of flux quantum per plaquette [20].

Since at $-9J_2 < J_1 < 0$ the nematic phase is characterized by a finite residual entropy $S_0 \approx 0.323$ [13], the first-order transition line separating it from the antiferromagnetic phase at finite temperatures is shifted to larger values of $|J_1|$ (in particular, at low temperatures it takes place at $J_1 \approx -9J_2 - 2S_0T$). Together with what we already know about the disordering of the antiferromagnetic and nematic phases this allows us to draw the schematic phase diagram for the case of a triangular lattice shown in Fig. 1(b).

Like in Fig. 1(a), curved arrows going from right to left show the evolution of the system with the decrease in temperature. The four arrows present in Fig. 1(b) correspond (starting from the lowest one) to scenarios F-N-AF, F-D-N-AF, F-D-AF, and F-D-C-AF, respectively. Here C denotes the phase with long-range order in chirality and vanishing superfluid density which separates AF and D phases at sufficiently high values of $T_0/J_2(T_0)$. Like for a bipartite lattice, the four-transition scenario (involving the path F-N-D-N-AF) is also possible if the region where $|J_1(T)|$ is comparable with T or smaller is sufficiently narrow.

Conclusion. – In the present letter we have investigated what happens with a phase-coherent array of SFS junctions when the decrease of temperature leads to the crossover of the junctions to the π-state. The corresponding phase diagrams have been constructed for the cases of a bipartite lattice (square or honeycomb) and of a triangular lattice. We have shown that the transition from the coherent phase existing well above the crossover to the coherent phase existing well below the crossover is never direct and these two phases are always separated by one or more intermediate phase(s). Naturally, the same approach can be used to construct the phase diagrams in the vicinity of the second crossover (from the π-state back to the conventional state) if it does exist. We hope that our results will stimulate more active experimental investigations of SFS junction arrays.

The Hamiltonian (1) can be also used for the description of a planar magnet with both bilinear and biquadratic exchange in the situation when the biquadratic exchange is ferromagnetic. For the case of the antiferromagnetic biquadratic exchange ($J_2 < 0$) such a system with a triangular lattice has been investigated by Park *et al.* [21]. Its phase diagram also includes a nematic phase, which however has a more complex structure of correlations (a three-sublattice one) than the nematic phase discussed in this work.

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